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Size in maximal triangle-free graphs and minimal graphs of diameter 2

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Dedicated to Paul Erdős on his eightieth birthday

Abstract

A triangle-free graph is *maximal* if the addition of any edge creates a triangle. For $n \geqslant 5$, we show there is an n-node m-edge maximal triangle-free graph if and only if it is complete bipartite or $2n-5 \leqslant m \leqslant \lfloor (n-1)^2/4 \rfloor + 1$. A diameter 2 graph is *minimal* if the deletion of any edge increases the diameter. We show that a triangle-free graph is maximal if and only if it is minimal of diameter 2.

For $n > n_0$ where n_0 is a vastly huge number, Füredi showed that an n-node nonbipartite minimal diameter 2 graph has at most $\lfloor (n-1)^2/4 \rfloor + 1$ edges. We demonstrate that $n_0 \geqslant 6$ by producing a 6-node nonbipartite minimal diameter 2 graph with 8 edges.

Keywords: Maximal triangle-free; Minimal diameter 2

1. Introduction

In general, we follow the notation and terminology of [5], the main exceptions being that the graph $G=(V,E)$ has $n=|V|$ nodes and $m=|E|$ edges. We also say G is an (n,m) graph and has *order n* and *size m*.

A *maximal triangle-free* (MTF) graph G has no triangles but the addition of any new edge creates a triangle. For example, the complete bipartite graph with node classes of orders a and b, here written $K_2[a,b]$, is MTF (see Fig. 1). Since it is connected, an n-node MTF graph has at least $n-1$ edges. Turán [9] showed that any n-node triangle-free graph has at most $\lfloor n^2/4 \rfloor$ edges. Since the n-node graphs $K_2[1,n-1]$ and $K_2[\lfloor n/2 \rfloor, \lceil n/2 \rceil]$ have $n-1$ and $\lfloor n^2/4 \rfloor$ edges, respectively, these bounds are sharp.

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However, for some values of n and m with $n-1<m<\lfloor n^2/4 \rfloor$, there are no n-node m-edge MTF graphs. For example, no MTF graph has 6 nodes and exactly 6 edges. Section 3 answers the question: *For which values of n and m are there (n, m) MTF graphs?*

The *diameter* of a graph is the maximum distance between any two nodes. A graph of diameter 2 *is minimal* if removing any edge increases the diameter. All the graphs in Fig. 1 are also minimal graphs of diameter 2 (MD2). Section 4 partially answers the question: *For which values of n and m are there (n, m) MD2 graphs?*

2. MTF graphs and MD2 graphs

Theorem 1 shows that all MTF graphs are MD2. Fig. 2 shows a MD2 graph that is not MTF. So the set of MTF graphs forms a proper subset of the set of MD2 graphs.

Theorem 1. *A triangle-free graph is MTF if and only if it is MD2.*

Proof (*necessity*). Let G be an MTF graph. If u, $v \in V$ and $d(u,v)>2$, then the edge uv could be added without forming a triangle. So the distance between any pair of nodes in G is at most 2 and thus G has diameter 2. Now suppose that an edge uv is deleted from G and the resulting graph $G-uv$ also has diameter 2. Then there must be a path

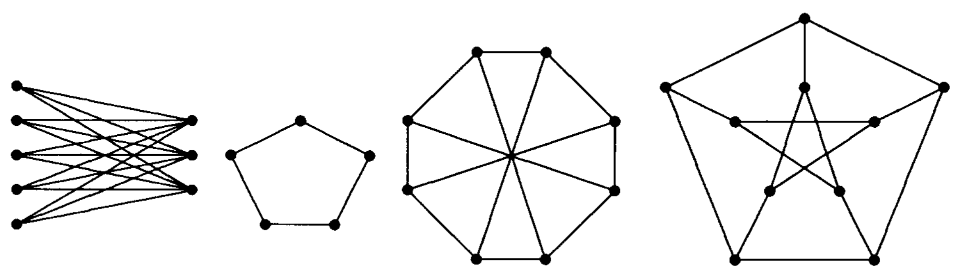

Fig. 1. Four MTF graphs: the complete bipartite graph $K_2[5,3]$, the 5-cycle C_5, the 8-node Möbius ladder [4] and the Petersen graph.

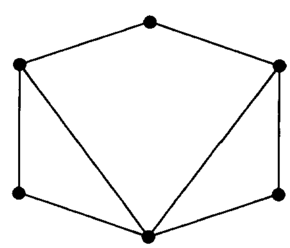

Fig. 2. An MD2 graph that is not MTF. Removing any edge from this diameter 2 graph results in a graph of diameter three or more, yet it has several triangles.

of length two between u and v in $G-uv$. This path together with uv forms a triangle in G. But G is triangle-free; so G is MD2.

Proof (*sufficiency*). Let G be a triangle-free MD2 graph. Suppose the edge uv is added to G. Since G is MD2 and there is no path of length 1 between u and v, there must be a path of length two between u and v in G. This path together with uv forms a triangle in $G+uv$. So G is MTF. □

3. The size of MTF graphs

For the purpose of this study, we need to define a family of nonbipartite MTF graphs which generalizes the 5-cycle C_5. Let p, q, r, s, and t be positive integers. The graph $C_5[p,q,r,s,t]$ on $p+q+r+s+t$ nodes has V partitioned into five subsets P, Q, R, S, T, with p, q, r, s, t nodes, respectively, having edges between all pairs of nodes in P and Q, in Q and $R, \ldots,$ and in T and P with no other edges (see Fig. 3). Then $C_5[p,q,r,s,t]$ has $pq+qr+rs+st+tp$ edges, is triangle-free, and is maximal because adding any edge creates a triangle.

The result in Theorem 2 follows from Murty [7, 8]. In [7] the Petersen graph is not on the list of extremal examples; [8] corrects this. Later Harary and Tindell [6] determined the structural characterization of the minimal 2-connected graphs of diameter 2. For completeness we present a new proof of the result here.

Theorem 2. *Every n-node 2-connected diameter two graph has at least $2n-5$ edges.*

Proof. Let $G=(V,E)$ be an n-node 2-connected diameter two graph with m edges. Let $\delta=\delta(G)$ be the minimum degree of G. Since G is 2-connected, $\delta \geqslant 2$. Also if $\delta \geqslant 4$, then $m \geqslant \delta n/2 \geqslant 2n$. So we are left to dispose of the cases when δ is either 2 or 3.

Case 1: $\delta=2$. Let $v \in V$ be adjacent to only w and x. Since G has diameter two, each node in $V-\{w,x\}$ must be adjacent to either w or x or both. Let $A \subseteq V-\{w,x\}$

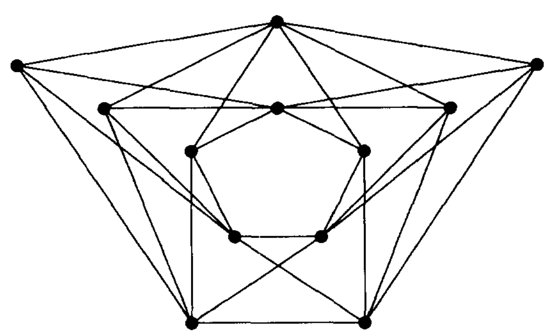

Fig. 3. The complete C_5-partite graph $C_5[2,3,2,2,3]$: an MTF graph with 12 nodes and 28 edges.

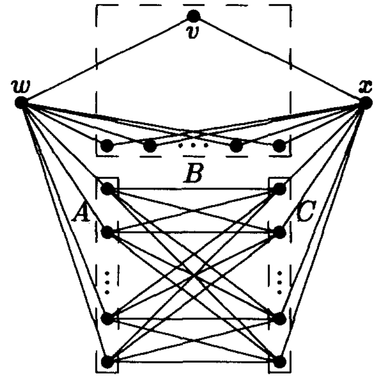

Fig. 4. The construction in Case 1 of Theorem 2. Node v is adjacent to w and x (w and x may or may not be adjacent). Each node (other than w and x) must be adjacent to w or x or both.

be the set of nodes adjacent to w but not to x. Let $B \subseteq V-\{w,x\}$ be the set of nodes adjacent to both w and x (note that $v \in B$). Let $C \subseteq V-\{w,x\}$ be the set of nodes adjacent to x but not to w. These three sets are illustrated in Fig. 4. Since A, B, and C partition $V-\{w,x\}$, we have $|A|+|B|+|C|=n-2$. Suppose $A=\emptyset$. Since G is 2-connected, the induced subgraph on $V-\{x\}$ must be connected and hence has at least $n-2$ edges. So

$$m \geqslant |E(x,B)|+|E(x,C)|+|E(V-\{x\})| \geqslant |B|+|C|+n-2=2n-4.$$

Similarly, if $C=\emptyset$, then $m \geqslant 2n-4$.

Suppose $|A| \geqslant 1$ and $|C| \geqslant 1$. Since each node of A has distance at most two from each node of C, the induced subgraph on $A \cup B \cup C$ must have a connected component that includes all nodes of $A \cup C$. Thus the induced subgraph on $A \cup B \cup C$ has at least $|A|+|C|-1$ edges. Hence,

$$\begin{aligned} m &\geqslant |E(w,A)|+|E(w,B)|+|E(x,B)|+|E(x,C)|+|E(A \cup B \cup C)| \\ &\geqslant |A|+|B|+|B|+|C|+(|A|+|C|-1)=2(|A|+|B|+|C|)-1=2n-5. \end{aligned}$$

Case 2: $\delta=3$. Let $v \in V$ be adjacent to only w, x, and y. Since G has diameter two, each node of $V-\{v,w,x,y\}$ must be adjacent to at least one of w, x or y (see Fig. 5). Counting three edges adjacent to v and at least one edge adjacent to each of the $n-4$ nodes in $S=V-\{v,w,x,y\}$, we find $d(w)+d(x)+d(x)+d(y) \geqslant n-1$. Since the number of edges equals half the sum of the degrees of the nodes,

$$m=\frac{1}{2}\left[d(v)+d(w)+d(x)+d(y)+\sum_{u \in S} d(u)\right] \geqslant \frac{1}{2}[3+n-1+3(n-4)]=2n-5. \quad \square$$

In the next theorem we completely answer the interpolation question for MTF graphs.

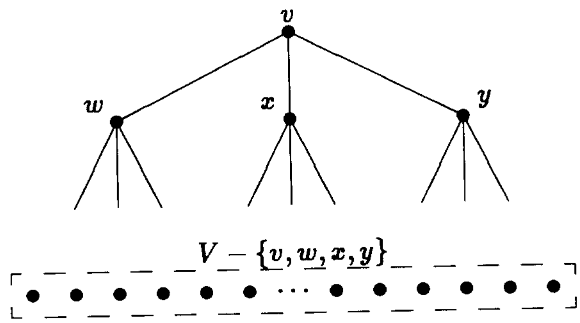

Fig. 5. The construction in Case 2 of Theorem 2. Node v is adjacent to w, x and y (these nodes need not be independent). Each node in $V-\{v,w,x,y\}$ is adjacent to at least one node of w, x, and y.

Theorem 3. *Let $n\geqslant 5$ and m be nonnegative integers. There is an (n,m) MTF graph if and only if $2n-5\leqslant m\leqslant\lfloor (n-1)^2/4\rfloor+1$ or $m=k(n-k)$ for some positive integer k.*

Proof (*necessity*). Let G be an (n,m) MTF graph. If G is bipartite, then it must be complete bipartite for otherwise an edge may be added without creating a triangle. Since $K_2[k,n-k]$ has n nodes and $k(n-k)$ edges, we have $m=k(n-k)$ for some k.

Now let G be a nonbipartite MTF graph. Since G is triangle-free and nonbipartite, it has at least one induced chord-free odd cycle C with length $k\geqslant 5$. Each node in $V-C$ can be adjacent to at most $(k-1)/2$ nodes in C for otherwise it would be adjacent to both nodes of an edge in C forming a triangle. Further since it is triangle-free, Turán's theorem [9] shows there are at most $\lfloor (n-k)^2/4\rfloor$ edges in the induced subgraph on $V-C$. Therefore,

$$m=|E(C)|+|E(V-C,C)|+|E(V-C)|\leqslant k+(n-k)(k-1)/2+\lfloor (n-k)^2/4\rfloor$$
$$=\left\lfloor\frac{(n-1)^2-(k-3)^2}{4}\right\rfloor+2\leqslant\left\lfloor\frac{(n-1)^2-(5-3)^2}{4}\right\rfloor+2=\left\lfloor\frac{(n-1)^2}{4}\right\rfloor+1.$$

Theorem 2 showed that every 2-connected MD2 graph has at most $2n-5$ edges. Since the only n-node 1-connected MD2 graph is $K_2[1,n-1]$ and Theorem 1 showed every MTF graph is MD2, we have that $m\geqslant 2n-5$.

Proof (*sufficiency*). It suffices to construct an (n,m) MTF graph for each permitted m. If $m=k(n-k)$ for some integer k, then $K_2[k,n-k]$ has n nodes and m edges. If $2n-5\leqslant m\leqslant\lfloor (n-1)^2/4\rfloor+1$, then an (n,m) MTF graph of the form $C_5[1,2,r,s,t]$ can be constructed as follows. Let $f(x)=x(n-x-5)$. Let s be the smallest positive integer with $f(s)\geqslant m-2n+5$. Since $m-2n+5\leqslant\lfloor (n-1)^2/4\rfloor+1-2n+5=\lfloor (n-5)^2/4\rfloor$ $=f(\lfloor (n-5)/2\rfloor)$, this is always possible. Note that if $m=2n-5$, then $s=1$ (and not 0). Thus $m-2n+5\geqslant f(s-1)$, or else s is not the smallest value. Let $t=f(s)-m+2n-4$. Then $t\geqslant m-2n+5-m+2n-4=1$ and $t=f(s)+1-(m-2n+5)\leqslant f(s)+1-f(s-1)$ $=n-3-2s$. Let $r=n-3-s-t$. Then $r=n-3-s-t\geqslant n-3-s-(n-3-2s)=s\geqslant 1$.

So r, s, and t are all positive integers, and hence $C_5[1,2,r,s,t]$ is an MTF graph with $3+r+s+t=3+(n-3-s-t)+s+t=n$ nodes and $2+2r+rs+st+t=2+(s+2)(n-3-s-t)+st+t=s(n-s-5)+2n-4-t=m$ edges. □

Theorem 3 shows that a list of the possible sizes of an MTF graph of order n has a "gap" between n and $2n-6$, and a series of gaps above $(n-1)^2/4+1$. For example, when $n=13$, such a list is 12, 21, 22, ..., 35, 36, 37, 40, 42. The corresponding graphs from the proof of Theorem 3 are $K_2[1,12]$, $C_5[1,2,2,1,7]$, $C_5[1,2,1,1,8]$, (and $K_2[2,11]$), $C_5[1,2,3,1,6]$, $C_5[1,2,4,1,5]$, $C_5[1,2,5,1,4]$, $C_5[1,2,6,1,3]$, $C_5[1,2,7,1,2]$, $C_5[1,2,8,1,1]$, $C_5[1,2,3,2,5]$, $C_5[1,2,4,2,4]$ (and $K_2[3,10]$), $C_5[1,2,5,2,3]$, $C_5[1,2,6,2,2]$, $C_5[1,2,7,2,1]$, $C_5[1,2,4,3,3]$, $C_5[1,2,5,3,2]$, $C_5[1,2,6,3,1]$ (and $K_2[4,9]$), $C_5[1,2,5,4,1]$, $K_2[5,8]$, and $K_2[6,7]$. There are other 13-node MTF graphs, but none have $0,1,\ldots,11,13,14,\ldots,20,38,39$ or 41 edges.

4. Minimal graphs of diameter 2

Simon and Murty (see [1]) conjectured that every n-node MD2 graph has at most $\lfloor n^2/4 \rfloor$ edges. Fan [2] proved the conjecture for $n \leqslant 24$ and $n=26$. Füredi [3] proved it for $n>n_0$ where n_0 is 'a tower of twos of height about 10^{14}'. He further proved that $n>n_0$, an n-node nonbipartite MD2 graph has at most $\lfloor (n-1)^2/4 \rfloor+1$ edges. It is interesting that this bound is the same as upper bound for the size of a nonbipartite MTF graph given in Theorem 2 even though the requirements for Füredi's bound are both weaker in that the graph need only by MD2 and stronger in that there must be a 'vastly huge number' of nodes. The $(6,8)$ graph in Fig. 2 shows that the condition $n>n_0$ is necessary and that the smallest possible value of n_0 is at least 6.

Füredi [3] proved that only a finite number of n-node nonbipartite MD2 graphs have more than $(n-1)^2/4+1$ edges. This is an example of such a "superdense" nonbipartite MD2 graph. An exhaustive search reveals that the only superdense nonbipartite MD2 graph with 10 or fewer nodes is the one shown above. So Füredi's finite number seems to be 1. Theorem 3 shows that a superdense nonbipartite MD2 graph cannot be triangle-free.

We may now use Theorems 1 and 3, and Füredi's result on nonbipartite graphs to give an analogue of Theorem 3 for MD2 graphs.

Theorem 4. *Let $n>n_0$ and m be positive integers. Then there is an (n,m) MD2 graph if and only if $2n-5 \leqslant m \leqslant \lfloor (n-1)^2/4 \rfloor+1$ or $m=k(n-k)$ for some integer k.*

We do not know of any superdense graph except the $(6,8)$ graph given in Fig. 2. Do there exist other n-node MD2 graphs with more than $\lfloor (n-1)^2/4 \rfloor+1$ edges?

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